Rewarding Duopoly Innovators: The Price of Exclusivity*

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Abstract

This paper considers the optimal design of patent rights for sequential innovations in an oligopoly. Firms contribute sequentially with unobserved efforts that transform ideas into improvements on a quality ladder. The optimal patent mechanism trades off incentives to encourage innovation efforts at different points in time. The optimal provision of incentives leads to strong asymmetries in the allocation of patent rights to firms and excludes all but one successful innovator in the limit. Treatment of the ex ante identical firms is ex post discriminatory. A simple implementation of the optimal mechanism with a system of patent fees is provided. Under an alternative policy arrangement, the allocation can be interpreted as optimal regulation of competition *for* the market.

1 Introduction

This paper studies the optimal reward structure for a sequence of innovations generated by firms who not small relative to the total volume of innovation. The appropriate reward for innovation has long been considered an important issue by economists. In describing the benefits of patents as a reward

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mechanism, [13] wrote that patents are an effective reward "because the reward conferred by it depends upon the invention's being found useful, and the greater the usefulness, the greater the reward." This paper builds on that general principle, that incentives dictate that rewards take the form of a stake in the profitability of the innovation. When innovations displace one another through a process of creative destruction, however, there is a trade-off: rewarding one innovation decreases the rewards to competing innovations. Often these competing innovations come from a few firms that interact repeatedly. We address the question of what forms of rewards best elicits innovation from those firms. Microsoft, for instance, argued that its strong position in the market was part of a sound policy in supporting innovation;[1] we address the trade-off between potential benefits from rewarding one firm greatly, and the cost that imposes in terms of lost innovation from other firms. We find that optimal rewards generate asymmetry, in the sense that the same opportunity receives greater rewards when it comes on the heels of other innovations by the same firm. Leads to a market which is frequently skewed towards a subset of the competing firms, leading to monopolization in the limit. The path to that limit can be alternately thought of as an optimal patent policy or as regulation of competition for the market.

Our approach is to study a class of constrained efficient allocations in a dynamic environment where innovations from the competing firms build on one another. Incentives for innovation are provided through the allocation of rights to sell products. Because of the cumulative nature of the research, allowing one innovator to profit in the product market necessarily restricts what can be offered to the other innovator, since they compete in this common market. As a result, rights are scarce. Our optimal reward structure allocates these rights across innovations to maximize total surplus.

In our model, firms contribute sequentially to a quality ladder. In order to make this contribution a firm must have an *idea* and make an *investment*. Ideas are random draws that arrive exogenously according to independent Poisson processes. Moral hazard arises with unobserved investments that turn these ideas into step increases on the quality ladder. The size of this step is a function of both the quality of the idea and the level of investment. We study an environment where there are no static monopoly pricing distortions, to focus on the trade off between rewarding innovations from the different innovators.

[1] See for instance [18].

Following the literature on sequential innovation, we assume that only one firm can profit from sales of the good representing this ladder at any point in time, while potential competition of others might limit its price. A *patent mechanism* allocates these rights to sell as a function of the history, in an incentive compatible way. Effectively, the mechanism grants the selected leader the right to exclude from competition any product that lags the current one by less than a certain number of steps. As an example, in the case of *forward exclusion* rights – considered by most of the previous papers and most closely resembling the existing structure of patent law – this interval includes only past contributions of the current leader, while no firm is excluded from competing with its own previous product developments.

A leading firm will profit from a past contribution to the ladder only at a time where that step is contained in its granted exclusion interval. In consequence, the marginal value of investment for a given step looks forward to all future instances where this step will be included in the exclusion interval. The larger is this set of instances, the higher the returns to investment and consequently the larger this step increase. This simple observation suggests two properties of an optimal patent mechanism that we prove in the paper. First, the exclusion interval should be *maximal*, because the larger the exclusion interval the more steps that are incentivized. So in absence of any other restrictions, this interval should include all steps, i.e. all other firms should be excluded from competing *in the market* even with their past product developments while the selected one captures all surplus. [2]

The second implication concerns a bias towards innovators that were more productive in the past, holding a lower bar for the quality of ideas to be implemented, a handicap in the competition *for the market;* identical opportunities are treated differently depending on the innovator's past history of contributions. While this unequal treatment is *ex-post* inefficient, it turns out to be *ex-ante* optimal as it incentivizes a larger set of prior innovation steps. In the extreme, this form of dominance will lead eventually to excluding forever from the patent race all but the luckiest innovator. Although such foreclosure is costly to the planner, the promise of foreclosure motivates innovation in the interim.

Our results are more stark when restricted to the case of forward exclusion – i.e. where firms are allowed to compete in the market with their own

[2]With the added restriction of forward exclusion, this set should include all past *continuous* contributions of the leading firm.

developed products – and there is no heterogeneity of ideas, which we analyze extensively in the paper. In this setting, the optimal patent policy displays a bang-bang property: there is no exclusion from new ideas coming from other innovators (i.e. no forward protection) until the incumbent reaches a threshold of n consecutive innovations, at which point all outsiders are excluded permanently from the innovation race.

Counter-intuitively, total exclusion occurs even when the marginal benefit of preferential treatment for the favored firm is zero at monopoly. The motive for this backloading of rewards is related to, but different from the standard backloading intuition from [4] and [12]. In those papers, backloading is beneficial not only because it generates strong incentives late, but also because it generates strong incentives early, as an agent exerts effort to increase the probability of reaching the point where the backloaded incentive kicks in. In our setting, the probability of reaching the backloaded state is fixed, since it is tied to the exogenous arrival of ideas. The planner backloads rewards because a unit of time allocated to the incumbent after several ideas provides incentives to increase the innovation step on all of those ideas simultaneously. This complementarity between number of innovations and the reward that the planner provides leads to protection whose duration increases with additional successes.

Another paper that finds that innovation rewards should be backloaded is [1]. They use a growth theory structure similar to [2] and compute numerically the best policies within a particular class, which includes both monetary rewards and exclusion rights. Their model has both the cumulative feature that leads to backloading here, as well as the "trickle down incentives" stressed in prior work on backloading with monetary rewards. We discuss below the differences between the two forces for backloading, which are both present in [1] but only one is present here. We prove the optimality of cumulative innovation backloading for a very broad set of state contingent policies. Although our structure does not nest theirs, the intuition about backloading that we develop applies to their environment. Numerically we show that the gains from backloading are higher the higher is the rate of innovation, suggesting that asymmetric policies are especially valuable in highly innovative areas.

We derive some additional implications for the *industrial organization* of innovation. Because the optimal patent mechanism exploits dynamic contracting opportunities, welfare increases when recurrence is more frequent. This occurs, for instance, when inventive ability is more concentrated: holding fixed the total arrival of ideas, welfare decreases with the number of innovators. Along the same lines, we show that welfare is higher when there is positive correlation in the arrival of ideas, where recent innovators are more likely to have follow on innovations than outside firms. Our results also suggest that in an environment where innovators differ in their *inventiveness* (measured by the rate of arrival of ideas), those that are slower can face a negative handicap and be held to higher standards for implementation.

Recent papers on optimal patents, beginning from [17], stress that ex ante heterogeneity across different innovation opportunities may lead to different rewards for different types of innovations. In our setting, even if there is no heterogeneity built into the structure, the rewards are history dependent, leading to ex post heterogeneity in the reward for different innovations.

We show that our optimal mechanism can be implemented through a system of non-infringing patents with an associated fee. The twist is that, since the optimal allocation forecloses the market, the most recent purchaser of a patent also has the right to pay an additional fee which disallows any more patents to be filed by the competition. With this decentralization, the authority need not observe anything, or ask for any reports; it simply allocates rights to anyone who pays the appropriate fees. Until the foreclosure fee is paid, the patent authority offers patents that are narrow, in the sense that they offer no rights to exclude other innovations; they simply give the innovator the right to exclusively market their own innovation. The foreclosure fee broadens the patent so that it excludes all future competition.

To complete our analysis, we consider the case of *total exclusion* in duopoly, namely where the selected firm is granted a monopoly right over the whole ladder. This corresponds to a patent pooling arrangement between the innovators where profits are generated by the entire history of innovations at every point in time. As pointed out earlier, total exclusion has an advantage in terms of incentives over forward exclusion, since each time a firm is in this position it receives a return on *all* previous innovations and not only on the last consecutive batch. There is an additional important difference to the forward exclusion case. Now the outsider may still have incentives to innovate even if by doing so it does not displace the leader *immediately*, for it will capture the returns to this innovation in all future instances where it is granted the monopoly right. The optimal mechanism assigns this right to firms contingent on the history of arrivals of ideas.

The results are qualitatively similar to the case where innovations compete; rewards are backloaded in the sense that one firm is nearly-permanently excluded in the long run. Median asymmetry between the firm, in a numerical example, lead to one of the innovators having a 50% higher claim on future profits than the less-preferred firm. Relative to the case where only forward exclusions are granted, welfare is as much as 50% higher compared than with only forward exclusions. These benefit of shared rights between firms are more pronounced the higher is the rate of innovation.

The allocation with complete exclusion can be interpreted as can be interpreted as regulating competition “for the market.”[3] The treatment strongly favors one firm until another firm generates a sufficient collection of innovations. The policymakers job, in this case, is to ensure that outcomes mimic the ex ante contract design, in cases where complete ex ante contracting may not be feasible.

Our paper links the literature on innovation rewards under asymmetric information with the literature that studies optimal protection in particular growth theory contexts. In addition to [17], papers in the former category include [6], who also generate a menu of patents for different types of innovations, and [10], where optimal policy is a menu of lengths and breadths. [9] and [14] apply these methods to dynamic environments, based on the quality ladder structure in [16]. In those papers the set of innovators is large, so there are never repeat innovators; they therefore can not address the issues of oligopoly, state dependent rewards, and the evolution of market structure that we study here. Like the model of [1], our paper allows us to study repeat innovators and their treatment as a function of their history of innovations. Unlike papers on innovation rewards that also feature asymmetric information like [11], [19], and [5], we do not consider the role of market signals in generating out optimal allocations.

The paper is organized as follows. In Section 2 we describe preferences, the production technology, and the innovation technology. In Section 3 we study optimal allocations when the planner is constrained to assign competing patent rights, without licensing. This environment most closely resembles existing papers in the patent literature and demonstrates our basic results, as well as a simple decentralization of the optimal allocation. In Section 4 we consider the case where that restriction is lifted, and interpret the results as a dynamic patent pool or competition for the market.

[3] In this sense our paper is related to the larger set of papers on regulation and innovation, for instance in [18] and [8]. [7] argue that competition for the market is as important as competition in the market.

2 Innovation and Competition

2.1 Static Competition

The quality ladder structure follows the one explored in the patent literature in papers such as [16] and [9]. We begin with a model of static oligopoly competition in a quality ladder. Suppose a collection of firms sells products of various quality levels. A single consumer[4] either takes an outside option (normalized to zero) or purchases one physical unit of the good of quality q that maximizes $q - w$, where w is the price paid for that variety.[5] There are no costs of production. We take competition to be Bertrand, so that the leading edge product is always the one sold in equilibrium, and the social surplus at any point in time is the quality q either in the form of profits for the firm selling the leading edge product, or as consumer surplus if $w < q$. The firm with the highest quality earns profits equal to the difference between the quality level of the highest and second highest quality level that is sold.

Our model abstracts from static monopoly costs. This highlights the role of the dynamic force that we study, namely the scarcity of rights when competition is for the market only, without any other source of inefficiency. We describe in several places how our results could be extended to the case with static distortions. Static distortions would naturally tend to decrease awards of monopoly rights relative to our solution.

2.2 Innovation

There is continuous time and an infinite horizon, with the future discounted at the rate r. We focus on the case where there are two agents (which we call firms or innovators) and a principal (or planner). Below we discuss extension to more firms. At any point in time a firm can be in either of two states: either it is endowed with a new *idea* or not. New ideas allow to produce a new quality level – an innovation – as described below. Endowment of an idea arrives to each firm at rate λ. [6]

[4] Or unit mass of identical consumers.

[5] As is usual in this sort of model, in the event of a tie, the higher quality product is chosen.

[6] Although we abstract from different λ across the innovators, nothing changes if λ differs across agents or differ for the firms based on which one had the last idea. We discuss this in the extensions section.

The innovation will be an improvement of size Δ upon the highest quality product currently available. Ideas can be turned (instantaneously) into innovations of size Δ in exchange for research cost $c(\Delta,\theta)$. The cost type θ of the idea is drawn from a continuous distribution $F(\theta)$. The stochastic process for the arrival of ideas and investment choices determine a path of arrivals $T_i = \{t_{i1}, t_{i2}, ...\}$ for player i and corresponding types $\Theta_i = \{\theta_{i1}, \theta_{i2}, ...\}$ and investment choices $\Gamma_i = \{\Delta_{i1}, \Delta_{i2}, ...\}$ .Denote by $T_i\,(t)$, $\Theta_i(t)$ and $\Gamma_i\,(t)$ the restrictions of these paths to arrivals prior to (and including) time t. Let $T\,(t) = T_1\,(t) \cup T_2\,(t)$, $\Theta(t) = \Theta_1(t) \cup \Theta_2\,(t)$ and $\Gamma\,(t) = \Gamma_1\,(t) \cup \Gamma_2\,(t)$. Correspondingly, at time t the sequence of innovations determines a frontier product $q\,(t) = \sum_{\Gamma(t)} \Delta_{in}$. Given that all consumers are identical, efficiency would require that this be the only product sold at time t.

2.2.1 Patent policy

A *patent policy* $\{\bar{\tau}_1\,(t)\,, \bar{\tau}_2\,(t)\} \in T\,(t) \times T\,(t)$ prescribes at time t the latest innovation each player is allowed to use. This patent policy can be identified also with exclusion rights comprising all those innovations that the leader (the firm entitled to the frontier product) can exclude the other firm from using. In the above case, the exclusion set is $\xi\,(t, \bar{\tau}_1, \bar{\tau}_2) = \{\tau \in T\,(t)\,|\bar{\tau}_2 < \tau \leq \bar{\tau}_1\}$. Profits are determined by Bertrand competition as above. For example, assuming $\bar{\tau}_1 > \bar{\tau}_2$ the price and profit flow obtained by the first innovator is $q\,(\bar{\tau}_1) - q\,(\bar{\tau}_2) = \sum_{\xi(t,\bar{\tau}_1,\bar{\tau}_2)} \Delta_\tau$. A *patent mechanism* prescribes a patent policy at time t as a function of the history of arrivals, i.e. a function from $T\,(t) \times \Theta(t)$ to the class of patent policies described above. The restriction to policies that are functions of the arrivals and not the actual innovations (the set $\Gamma\,(t)$) is common to most of the literature on sequential innovation and is the fundamental source of moral hazard. As in papers including [17] and [9], were Δ observable and contractible, the optimal contract would be to reward based on cash prizes based on Δ, rather than scarce market time.

The patent mechanism implies that only one firm can earn profits at any instant. Moreover, profits are granted through an exclusive right to exploit a subset of the given innovations; these rights can only depend on the sequence of ideas and cost types but not the size of the innovations. We describe the problem assuming that arrivals are public information, but our implementation discussed in Section 3.3.2 shows that the planner need not observe the arrival of ideas, and therefore this assumption is without loss of generality.

It might appear that a patent policy would induce strategic interaction between the firms. Under the additive structure in the quality ladder, however, the impact of any firm's level of innovation on payoffs, given the policy, is separable from the innovation level of the other firm, and therefore we can treat the decisions separately. The only link comes through the history-dependent promises generated by the policy. We describe this in more detail for the two classes of policies we study. In the forward exclusion case the payoffs are not only separable but in fact independent.

2.2.2 Incentives for innovation

We now derive the incentives for innovation from a given patent mechanism. Take for example an idea obtained by player one at time τ_i. Player one will be able to profit from this innovation every period where $\bar{\tau}_2(t) < \tau_i \leq \bar{\tau}_1(t)$ In every such period, player one will receive a flow of payments Δ_i for this particular innovation. Letting

$$d_i = E \int_{\bar{\tau}_2(t) < \tau_i \leq \bar{\tau}_1(t)} e^{-r(t-\tau_i)} dt$$

this gives player one an expected discounted value from this innovation equal to $\Delta_i d_i$. Because of the strong separability of the innovations' contributions to quality, this is all the relevant information a firm needs to know in order to choose optimally its investment to transform ideas into innovations. We refer to this expected discounted time during which the innovator profits from the current innovation as *duration.*

When innovator i chooses Δ for a particular value of d, he solves

$$\Delta(d, \theta) = \arg\max_{\Delta} d\Delta - c(\Delta, \theta)$$

The features of the contract, for the purposes of the investment decision, can be summarized by the planner's promise of duration $d \geq 0$ during which the innovator will be given preferential treatment for an innovation made under that idea. This simplification is a key feature that allows the complete contingent rights contract to be tractable in a recursive way we introduce below. We assume that higher θ corresponds to a greater impact of duration on innovation:

Assumption. $\Delta_{12} > 0$

This assumption corresponds to a third derivative condition on c, which is satisfied, for instance, $c(\Delta, \theta) = \Delta^\alpha/\theta + k$ with $\alpha > 1$.

A particular duration d for a given innovation can be granted in many different ways. For example, a T period patent (where preference is guaranteed for all T periods) would have $d = (1 - e^{-rT})/r$. A patent that offered T periods of protection for sure, followed by T' units of additional protection with probability 1/2 would have $d = (1-e^{-rT})/r + \frac{1}{2}e^{-rT}(1-e^{-rT'})/r$. Since the planner can choose the allocation of rights at every instant, this duration can evolve deterministically over time, change with later arrivals of innovators, and with the identity and type of those innovators that arrives with an idea. Because discounted time is not unbounded, the maximum possible promise of sure preferential treatment forever is $1/r$. This bound highlight the fact that rights are a limited resource that must be allocated in the most efficient way to provide incentives for innovation by firms. We use the language of duration to describe recursively how the optimal policy proceeds, considering arbitrary duration policies, which may be contingent on future arrivals as well as the passage of time.

The strong separability of innovations highlighted before carries over to the calculation of social value. Since any innovation Δ contributes this increment from it inception into the infinite future, it yields $\Delta/r - c(\Delta, \theta)$ additional units of present discounted social surplus.The social contribution of allocating d units of duration to an idea θ is then $R(d, \theta) = \Delta(d, \theta)/r - c(\Delta(d, \theta), \theta)$ Note that the derivative with respect to duration is

$$\begin{aligned} R_1(d, \theta) &= \Delta_1(d, \theta)/r - c_1(\Delta(d, \theta), \theta)\Delta_1(d, \theta) \\ &= \Delta_1(d, \theta)(1/r - d) \geq 0 \end{aligned}$$

where the second line uses the fact that $c_1(\Delta, \theta) = d$ by the innovator's FOC. Since convexity of costs implies that $\Delta_1(d, \theta) > 0$, the return is strictly increasing in duration except where $d = 1/r$. Our model has no static distortions, so that $R(d)$ is maximized at $1/r$. Modifying the model so that $R(d)$ is not maximized at $1/r$ can be interpreted as allowing for static costs of monopoly, and would not undo the message of backloading. Under the sorting condition $\Delta_{12} > 0$, the marginal return to duration increases in type, $R_{12} > 0$.

In order to maintain concavity of the planner's problem, we assume

Assumption. $R(d)$ is concave.[7]

The stock of available rewards available to a particular innovator is curtailed by the amount that is promised to the competition. The allocation of duration to the current innovator, which generates value via R, against the cost generated by the promise of future duration. We now turn to studying the optimal allocation of duration across histories.

3 Forward Exclusion Rights

3.1 Policy Space

In this section we consider a restricted set of contingent rights that is closest in spirit to a typical view of a patent in many dynamic models of patents such as [16]. In particular, the planner offers innovators exclusive rights to market their own innovation, and potentially excludes some (small) improvements to the patented work. Innovators maintain their rights even after they are superseded. We call this protection *forward exclusion rights*, and it is equivalent to forward patent breadth in the language of [16]. It most closely matches most of patent structures, where a future innovation does not exclude a past innovator from producing something they have already invented. The planner, however, can potentially benefit from offering such backward exclusions, which we study below.

Let $q_i(t)$ denote the product that this policy entitles innovator i to sell. (More formally, letting $\tau_i(t)$ denote the last time player i innovated before t, $q_i(t) = q(\tau_i(t))$.) Assuming for the sake of argument that player 1 is the one with the last innovation, $q_2(t) < q_1(t) = q(t)$, Bertrand competition will entitle players profit flows $\pi_1(t) = q_1(t) - q_2(t)$ and $\pi_2(t) = 0$. This patent right that is non-infringing on past rights, but does not encompass the prior rights. The most recent patent holder makes profits equal to the

[7] By the implicit function theorem it must be the case that

$$\Delta'(d) = \frac{1}{c''(\Delta)}$$

We then have that

$$R''(d) = \Delta''(d)(1/r - d) - \Delta'(d)$$

In order for R to be concave, then, we need a joint condition on the second and third derivatives of c. A sufficient condition is that $c''' > 0$.

gap between his quality and the quality level of the most recent innovation by his rival.[8] In this case the actions of the other firm have no impact on the payoff for a firm considering innovation, as the timing of arrivals of the opponent is exogenous, and the profitability of the leader is independent of the past innovation decisions of the laggard firm.

It is instructive, before considering the recursive structure of the problem, to think about the limits of the planner's powers to offer exclusion rights. On the one hand, the planner could offer "full exclusion" to the most recent innovator, excluding all of the competition's future ideas. Such duration would imply $d = 1/r$ for all of the incumbents innovations forever, but get nothing from the other firm. On the other hand, if the planner promised no forward exclusion of future innovations, each innovation would receive duration $d = 1/(r + \lambda)$, i.e. the discounted time until the next arrival by the other innovator. Duration $d \leq 1/(r + \lambda)$ can be delivered without excluding anything, and there is scarcity in duration since $1/(r+\lambda) < 1/r$, which is the amount of duration required to implement the first best level of innovation.

Clearly, if patent holders could contract on Δ, they would like to buy and sell patents based on their size. This would undo the informational rationale for patents at the heart of [17] and this paper. We therefore do not allow such transactions. However in the next section we study complete exclusion rights, which have the interpretation as being a licensing contract signed ex ante between the firms, under the restriction that the contract cannot condition of Δ.[9]

3.2 Dynamic Program

We now turn to the optimal choice of duration. We solve the problem recursively. The planner has an outstanding promise of d as a result of commitments made previously to the current incumbent. If this incumbent receives a new arrival θ ,the planner grants an updated duration $d_1(\theta)$. Alternatively, if the non-incumbent receives an arrival θ, the planner can either choose

[8] In Hopenhayn et al. (2006), a patent system with a single rights holder at any instant is defined to be exclusive. We avoid that terminology to avoid confusion with the rights described in the next section, where at any point in time one innovator has exclusive rights to the entire ladder.

[9] In this section the optimal allocation can be interpreted as a licensing contract where firm can never commit to give up rights to sell products that they invented previously; i.e. they could sell licenses but not sell away their own right to produce.

not to implement the innovation (offering it duration zero, and maintaining the current incumbent's position), or to implement it by offering some $d_0(\theta)$, making the outsider the new incumbent.[10] It is immediate that, among excluded ideas, the planner always chooses to exclude the worst ones; therefore there is some cutoff rule $\bar{\theta}$ below which ideas of the non-incumbent are excluded. The planner's problem, written in terms of the planner's value function V, is

$$\begin{aligned} rV(d) &= \max_{d_1(\theta), d_2(\theta), \bar{\theta}} \left\{ \begin{array}{c} \lambda \int \left(R(d_1(\theta), \theta) + V(d_1(\theta)) - V(d)\right) dF(\theta) + \\ \lambda \int_{\bar{\theta}}^{\infty} \left(R(d_2(\theta), \theta) + V(d_2(\theta)) - V(d)\right) dF(\theta) \end{array} \right\} \\ & \quad s.t. \\ rd &= 1 + \lambda \left(\int d_1(\theta) dF(\theta) - d \right) - \lambda \left(1 - F(\bar{\theta})\right) d \end{aligned} \tag{1}$$

where d, d_1, and d_2 can be taken to lie in $[1/(r+\lambda), 1/r]$, since if $d < 1/(r+\lambda)$, the planner always could have offered the prior innovator more time without increasing exclusions, and therefore increase welfare.

This most recent incumbent has *forward* exclusions that guarantee a duration d for his state-of-the-art product. Future arrivals of the other firm may be excluded in order to deliver d.[11] The constraint in (1), which we call promise keeping (PK), guarantees that the planner actually does deliver d and is critical to understanding the problem. Given a current promise d, the planner delivers intervening instants until the next arrival. If the next arrival is by the incumbent, his duration promise increments to d_1; if the next arrival is by the other firm, the incumbent is supplanted with probability $1 - F(\bar{\theta})$, i.e. if the arrival is good enough to not be excluded. The constraint simply says that, in expectation, the duration offered delivers the promise of d. This constraint is what allows us to solve the fully history dependent problem of allocating rewards in a tractable, recursive way.

This constraint also shows a key difference between this model and one with a sequence of innovators, as studied in Hopenhayn, et al (2006). In both models, duration promises to the current innovator make the PK constraint

[10] In the appendix we verify that the planner neither wants to adjust the incumbents promise when an arrival occurs that is not good enough to be implemented, nor at instants when nothing arrives.

[11] For now, the interpretation is that no further contracting, such as licensing, is possible. In the next section we allow for the possibility that the innovators pool their patented work and share the returns according to a pre-specified rule that can be interpreted as a rich ex ante licensing agreement. None of the key features of the model or the results differ.

tighter in the future. In simple terms, increasing duration today makes the planner less able to make promises to other agents in the future. However, to the extent that future innovations come from the same source, greater duration does not preclude future innovations, and therefore is *not* making the PK constraint tighter in the future in those states. This is the fundamental difference between this problem from and one where innovators never recur. From the promise keeping constraint we see the tradeoff between offering rewards d_1 to the current innovator, and implementing outsiders: the greater is the reward to the insider, through a promise of d_1 for their next innovation, the more outsiders are implemented.

We characterize the solution to the dynamic programming problem using first order conditions.[12] The first order condition for $d_1(\theta)$ is

$$R_1(d_1, \theta) + V'(d_1) = \mu$$

where μ is the Lagrange multiplier on the PK constraint. The envelope condition for the value function implies that

$$\mu = V'(d)$$

Since $R_{12} > 0$, substituting for μ in the first order condition immediately implies that the policy function $d_1(\theta, d)$ (i.e. the choice of duration as a function of the type and current state) has the following properties.

Lemma 1. *$d_1(\theta, d)$ is weakly increasing in θ and d. Moreover, $d_1(\theta, d) > d$.*

Better arrivals are granted more duration and new promised duration is increasing in the outstanding promise. It also follows immediately that the planner never excludes innovations by the incumbent, since there is no trade-off between rewarding new innovations by the incumbent and keeping the promise on old innovations. In this sense the incumbent is immediately favored relative to the outsider, who faces some exclusion. The second part of the Lemma implies that duration rises with each new arrival of the incumbent, so with a series of consecutive arrivals d cannot converge to a level less than $1/r$. Therefore the allocations eventually have near monopolization, in the sense that eventually the system evolves to a point where one firm is promised almost the entire future. A firm with many consecutive successes

[12] In the appendix we verify that the value function is concave and differentiable.

will come to dominate the market, nearly forever. Correspondingly, almost all ideas of the other firm will not be developed.

We assume that θ is observable. However, in papers such as [9], monotonic policies are shown to be implementable through an appropriate fee system; larger rewards are given greater protection but have to pay a higher fee. Here such a decentralization would need to depend both of θ and the firm's incumbency status. We focus on the latter part of the decentralization, how it depends on the firm's incumbency, in the next section, where we take all ideas to be of the same type.

3.3 Homogeneous Ideas

In this section we show that, without heterogeneity of cost types, the monopolization of the market happens in finite time. We demonstrate the sense in which the model generates heterogeneity in innovation size endogenously; all ideas are identical, but differential treatment leads to heterogeneous improvements. We develop the result in a way that explains further the mechanism behind the backloading in the model. Since all ideas are identical, we suppress the θ variable.

3.3.1 Backloading

The absence of heterogeneity implies that in order to deliver on the promise d, the planner may need to mix between implementing and not implementing a new idea from the non-incumbent. Denoting the probability of implementation of an idea by the non-incumbent firm by p the dynamic program is

$$\begin{aligned} rV(d) &= \max_{d_1,d_2,p} \left\{ \begin{array}{c} \lambda\left(R(d_1)+V(d_1)-V(d)\right)+ \\ \lambda p\left(R(d_2)+V(d_2)-V(d)\right) \end{array} \right\} \qquad (2) \\ & \quad s.t. \\ rd &= 1+\lambda(d_1-d)-\lambda pd \end{aligned}$$

Note that, from the perspective of the incumbent, p plays the same role as $1-F(\bar{\theta})$ in the general case.[13]

Lemma 2. *There exists* $\widehat{d} \leq \frac{1}{r}$ *such that* $p=1$ *if* $d \leq \widehat{d}$*, is strictly decreasing and positive when* $\widehat{d} < d < \frac{1}{r}$ *and equal to zero when* $d=\frac{1}{r}$*.*

[13]When omitted in the text, proofs are contained in the appendices.

Proof. First note that the choice of d_2 is independent of d and satisfies $R'(d_2) + V'(d_2) = 0$. As for the choice of p it is either interior, in which case $R(d_2) + V(d_2) - (V(d) - \mu d) = 0$ (where μ is the multiplier of the promise-keeping constraint) or equal to zero or one. By the envelope condition $V'(d) = \mu$, so the term in brackets equals $V(d) - V'(d)\, d$. This term is increasing in d and can only be constant in a region where $V'(d)$ is constant. From the first order conditions for d_1 it follows that when $\mu\,(= V'(d))$ is constant, d_1 is also constant and hence p must decrease with d. Also note that the fact that p is decreasing implies that when zero, it must be the case that $d = \frac{1}{r}$. Hence there exist $\widehat{d}$ such that $p = 1$ if $d \leq \widehat{d}$, is strictly decreasing and positive when $\widehat{d} < d < \frac{1}{r}$ and equal to zero when $d = \frac{1}{r}$. □

Proposition 3. *If* $d_1(d) < 1/r$ *then* $p(d) = 1$ *and if* $p(d) < 1$ *then* $d_1(d) = \frac{1}{r}$.

This proposition provides a bang-bang result: either there is **no** exclusion for the outsider or there is **maximal exclusion** (upon one more arrival of the incumbent.) The intuition for this result can be given with aid of Figure 1. Consider an incumbent with promise d. In the figure, $p_0 = p(d)$ and $p_1 = p(d_1(d))$. If the outsider has an innovation, with probability p_0 it is implemented and the duration promise goes to zero. If the incumbent gets an idea before the next time an outsider is implemented, the duration promise updates to $d_1(d)$ and the planner revises the probability of an outsider being implemented to p_1. The intuition behind this result is as follows. Suppose towards a contradiction $p_1 > 0$ and $p_0 < 1$. From Lemma 2 it follows immediately that $d_1 < \frac{1}{r}$. Now consider a variation where p_0 and d_1 are increased slightly, maintaining the original value d. This has no impact on the initial incentives for investment as d is unchanged; it increases the incentives to innovate upon the next arrival as d_1 is increased. The key is that it also generates the same return from outside innovations: since d is the same, it requires the same expected discounted exclusions of the outsider. Since the revised policy gives the same initial innovation and expected exclusion cost as the original plan, together with more innovation incentives for the next arrival, it thus increases total value. [14]

This argument makes clear the intuition behind backloading, and how it is specific to the patent context. Since the planner is committed to d, he is committed to a fixed amount of exclusions of the non-incumbent firm.

[14] In conjunction with the previous Lemma, this argument shows that there can be at most one period where $0 < p < 1$ and the Proposition easily follows.

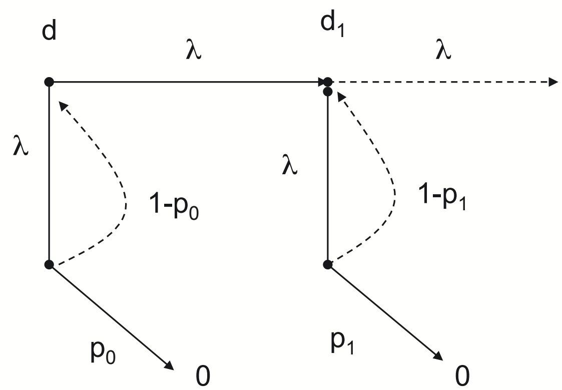

Figure 1: Optimal exclusions

The timing of the exclusions is welfare neutral, in the sense that all of the exclusions cost the planner missing out on a new incumbent starting with d_2. When the planner implements duration d by excluding arrivals of the outside firm later, he raises the duration promise for all intervening arrivals by the incumbent, which raises the incumbents level of innovation. In other words, a scarce unit of time allocated later rewards both early and late innovations.

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The above result can be also used to easily construct the path for duration starting from any initial value d. Note that for any given p, the promise keeping constraint gives a linear relationship between d_1 and d. Figure 2 shows this relationship for $p = 0$ and $p = 1$. First note that for $p = 1$, $\underline{d} = \frac{1}{\lambda+r}$ is a fixed point. This corroborates that this natural rate of duration is granted without any exclusions. For any higher initial value of d, the sequence of durations generated upon additional arrivals is strictly increasing. Note also that any durations to the right of $\widehat{d}$ require $p < 1$. The dynamics for optimal duration is thus the increasing sequence generated by the upper line until a value $\widehat{d} \leq d \leq \frac{1}{r}$ is reached, where p is chosen so that next period duration is $\frac{1}{r}$.

This path assumes the initial level of d is above the natural rate $\frac{1}{\lambda+r}$, which is proved in Lemma 4.

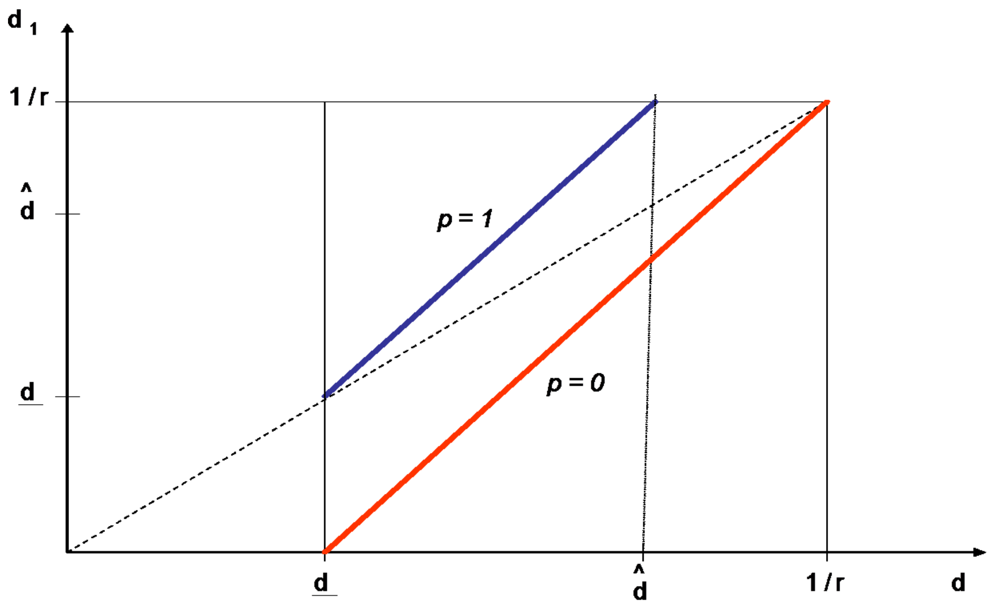


Figure 2: Evolution of duration

Lemma 4. $d_2 > 1/(r + \lambda)$.

The Lemma shows that new incumbents are promised some future exclusion of their competitor. The earlier results show that these exclusions are maximally backloaded, which generates duration promises that climb, with positive probability, to $d = 1/r$.

The intuition behind offering some exclusions to new incumbents is related to backloading. If the planner were forced to grant exclusions that generate d_2 immediately (i.e. for ideas that arrive immediately after the change in incumbency), then exclusions would not be beneficial and d_2 would be exactly $1/(r + \lambda)$. The reason is concavity of R: immediate exclusions cost $R(d_2)$ in lost innovations, but generate $R'(d_2)$. The latter is always smaller when R is strictly concave, but would be identical if R were linear. However, backloading leaves the cost of exclusions the same, but increases their benefit: exclusions far in the future generate benefits for all of the incumbents ideas that arrive in sequence in the meantime. The proof extends this logic to show that for any concave R, a small amount of exclusions, sufficiently backloaded, is beneficial to the planner.

The following Proposition summarizes the results developed above.

Proposition 5. ***(Optimal patent policy)*** *There exist an integer N and $p_{N-1} \in [0,1]$ such that an arrival of the outsider is implemented for sure iff it occurs before $N-1$ consecutive arrivals of the incumbent; it is implemented with probability zero if it comes after N consecutive arrivals of the incumbent and with probability p_{N-1} if it comes after $N-1$ consecutive arrivals.*

Backloading of rewards is similar to the quantitative result in [1]. They stress the usual backloading motive as in [4] and [12] when they describe "trickle down incentives." Since in their model the *probability* of innovation is tied to effort, backloaded rewards encourage agents to work hard to visit the backloaded state. Our model shows a different reason for backloading. The driving force is not to encourage innovators through the promise of a large prize after N successes, as arrivals are exogenous. In our model it is the cumulative nature of innovations that plays a central role. To clarify the difference, consider a restricted set of patent mechanisms that allows the outsider to use all but the last step of the incumbent, so that after any sequence of consecutive innovations, the incumbent will derive profits only from the last step. With this restriction, the optimal policy will have no backloading and the outsider's ideas will never be excluded. For backloading to arise, it is central that exclusions that occur after several arrivals reward the incumbent for all interim innovation steps. Our results therefore clarify [1], where both this force and the more standard motive for backloading are present.

As shown in the Appendix the derivative of the value function has a simple expression:

$$V'(d) = \sum_{n=1}^{N} R'(d_n) - r\left[R(d_2) + V(d_2) - V\left(\frac{1}{r}\right)\right], \tag{3}$$

where N corresponds to the exclusion period and $d_n$is the sequence generated using the promise-keeping constraint starting from d. The two components in this formula highlight the benefits and costs of higher initial duration. The benefit is that higher current duration allows to provide more duration for future arrivals and consequently additional value. On the other hand, higher duration increases the probability of exclusion and this carries a cost as measured by the term in brackets.

Equation (3) can be used to show the value of exclusion. In absence of exclusions, at the natural rate $\underline{d}$ where N is infinite, the formula shows that

$V'(\underline{d}) = \infty$, as all terms in the series are identical and positive. Finally note that strict concavity of the value function follows immediately from equation (3). [15]

3.3.2 Implementation

Next we turn to the question of how this allocation could be implemented by the planner when arrivals are private to the firms. Our implementation also requires minimal information on the side of the firms. To simplify our exposition we focus on the special case where $p_{N-1} = 0$ so there is no exclusion until the N^{th} arrival from the incumbent and full exclusion afterward. This simplifies the necessary tools to implement the optimal allocation, and shows the general principal behind the implementation. The planner employs two fees $\{f, t\}$. The first is a typical *patent fee* paid by an outsider who wants to become the new incumbent, and entitling the holder to a non-infringing patent on their innovation that lasts forever, but offers no "forward" protection: it effectively ends when another innovator pays the same fee and takes the lead. The second one is an *exclusive rights fee*, which can be paid at any time by the current leader (i.e. the most recent payer of the patent fee), and which changes rights in one fundamental way: it disallows the competition from ever being granted a patent in the future. In essence it generates infinite forward breadth, which forecloses the market, since licensing is effectively impossible given moral hazard and the assumption of incomplete exclusion.

We consider implementation in sequential equilibria. To define the game, take the following information structure at time t. Both firms know who is the incumbent $I \in \{i, j\}$ and whether exclusion has occurred or not. In addition, the incumbent knows the number of consecutive arrivals n since displacing the outsider. As shown in the appendix, it is sufficient to consider strategies at a restricted set of nodes. These sets are defined by the times of new arrivals for each player. For the outsider, the strategy upon an arrival is the entry decision: whether to displace the incumbent and pay the fee f or not. An incumbent decides after each arrival $n \geq 1$ the step size Δ_n and whether to pay the termination fee t. These strategies, together with the fees (f, t) define payoffs for the players in the natural way. The problem is to find a pair of fees (f, t) that implement the strategies *enter* for the outsider

[15] In addition, it also follows that V is differentiable at all points including those where N changes, as the terms dropped or added have $d_N = \frac{1}{r}$ and thus $R'(d_N) = 0$.

and termination at N for the incumbent, together with the sequence of steps $\Delta_1, ..., \Delta_N$ that form a sequential equilibrium. Note also that the strategies can be reduced to two main decisions: for the outsider the entry decision and for the incumbent the number of arrivals M before paying the exclusionary fee; the optimal step increases for the incumbent follow immediately from these decisions.

Proposition 6. *There exists fees (f, t) that implement the optimal patent system as a sequential equilibrium. That is, an outsider pays f only upon receiving an idea and the incumbent pays t after exactly N arrivals.*

The implementation can be interpreted as a screening device. The way in which screening of the outsider is obtained is standard: the value of taking over leadership just covers the patent fee f, and therefore anyone without an innovation does not find it worthwhile to claim to have an innovation when they do not. As for the incumbent, screening on the number of consecutive innovations works because the greater is the number of steps that an innovator is profiting from, the higher is the value of exclusivity. Therefore the fee needs to be set sufficiently high that only an innovator with at least N arrivals will be willing to pay the fee.One can interpret t as the "price of exclusivity," a fee paid to make the firm never face competition from additional innovations.

To understand the role the fee plays, consider an alternative structure where the planner auctions, at each instant, patent rights that grant the winning bidder the right to exclude the loser from using any of the winning's most recent series of innovations. With equal footing for the two firms, the outsider would never have the incentive to innovate, since the profitability of a new innovation that competes with an old one is always the lower than for any other innovation, so the incumbent would always win the subsequent auction.[16] The optimal rule would need to handicap such an auction in favor of the outsider in order to delay monopolization. The handicap or price of exclusivity trades off the incentive benefit of backloaded rewards that come with eventual exclusivity against the fact that, left to decide on their own, the firms would choose exclusivity too soon in order to limit competition.

[16]Such an auction would succeed at maximizing joint profits for the two firms, by focusing all innovation at one firm, but at the expense of consumer surplus. While exclusivity is inevitable in the socially optimal allocation, the firms' joint interests would be served if exclusivity arrived immediately, since outsider innovation always reduces profits.

3.3.3 Different arrival rates or more than two firms

In this section we consider a situation in which the incumbent has an arrival rate that is different from the outsider. The same logic and results apply to the case where there are more than 2 innovators, as the outsiders' total arrival rate is the sum over individual arrivals. To focus on heterogeneity and not on the effect of differing aggregate innovation opportunities, our analysis maintains the sum of arrivals constant and equal to λ, where $\alpha\lambda$ corresponds to the incumbent's recurrence rate and $(1-\alpha)\lambda$ to the sum of outsiders' arrivals. As an example, if the innovation ladder comprises M firms with equal arrival rates, then $\alpha = 1/M$.

The dynamic program is

$$\begin{aligned} rV(d;\alpha) &= \max_{d_1,d_2,p} \left\{ \begin{array}{c} \lambda\alpha\left(R(d_1)+V(d_1;\alpha)-V(d;\alpha)\right)+ \\ \lambda\left(1-\alpha\right)p\left(R(d_2)+V_M(d_2)-V_M(d)\right) \end{array} \right\} \\ & \quad s.t. \\ rd &= 1+\lambda\alpha(d_1-d)-\lambda\left(1-\alpha\right)pd \end{aligned} \tag{4}$$

This problem is very similar to the one described above; indeed, the characterization of section 3.3.1 applies directly, so that p is maximized until $d_1(d) = 1/r$. In particular, in the application to many firms, no competitors are excluded until all are, and a decentralization like the one provided above can still be used. We prove two facts regarding comparative statics in α. First:

Proposition 7. *$V(d;\alpha)$ is increasing in α.*

The intuition is straightforward and can be easily explained for the case $\alpha = \frac{1}{M}$, where concentration of ideas (lower number of firms M) benefits the planner by allowing better dynamic contracting opportunities. We describe the intuition more precisely after we characterize the comparative statics on d_2. To derive these we first show that the marginal cost of allocating duration decreases with α (increases with the number of firms M)

Proposition 8. *Take $d' > d$. $V(d';\alpha) - V(d;\alpha)$ is increasing in α.*

This is intuitive: the marginal cost of allocating duration is $\partial V(d;\alpha)/\partial d$, which decreases when the current promise holder is more likely to be the next innovator. This implies that duration d and recurrence rate α are complements and given that the optimal initial duration d_2 maximizes the sum $R(d_2)+V(d_2;\alpha)$, it follows immediately that d_2 increases in α.

Corollary 9. *d_2 is increasing in α.*

While this corollary states that initial duration increases with α, this does not necessarily mean that there will be more exclusions, in the sense of reaching full exclusion sooner. For a fixed exclusionary policy (N, p) initial duration increases with α as it becomes more likely and faster for the incumbent to reach the exclusion region. However, we can establish that exclusion must increase in α for very low levels of recurrence. As α approaches zero (M approaches infinity), there is no recurrence so every innovator is unique. In that case there are no gains from backloading and the optimal policy approaches one that offers every idea the natural rate $d = 1/(r + \lambda)$, with no exclusion of others' ideas. But for higher α (smaller M) we have proved that exclusions are optimal.

Exclusions arise because of two ways they benefit the planner. First, when the same innovator has consecutive innovations the planner can allow the innovator to profit from both, so the allocation of duration to the second innovation does not compete away duration of the first. Moreover, there are benefits from backloading and those benefits are only achievable to the extent that innovators recur. The more firms are competing, the less is offered to a new incumbent, and therefore the less likely is monopolization by that incumbent. Monopolization is still inevitable, but slower since the cost of exclusion is increasing in M.

Tables 1 and 2 provide numerical results for a set of arrival rates $\lambda \epsilon \{0.1, 1, 12\}$ with expected time of arrivals of 10 years, 1 year and 1 month, respectively and incumbent recurrence rate $\alpha \in \left\{0, \frac{1}{8}, \frac{1}{2}, \frac{9}{10}\right\}$. In addition to these parameters, the interest rate $r = 0.05$ consistent with a yearly time unit and the cost function quadratic. In Table 1, for each value of λ, the first column gives our measure of duration, the second column $N.p$, the number of periods prior to exclusion of entrants plus the probability of exclusion thereafter and the third column the probability that entrants will be excluded, calculated at the beginning of incumbency. For fixed λ, duration increases with α (as predicted by the theory), though it takes more arrivals to exclude the rival. The intuition behind this result is that when the rate of recurrence is high, the probability of being displaced is lower conferring the incumbent a higher natural barrier and protection. The combined effect is much higher probabilities of exclusion. An increase in λ reduces the natural rate $1/(r + \lambda(1 - \alpha)$ for every α. To mitigate this reduction, the number of periods to exclusion are decreased leading at the end to a higher total probability of exclusion.

Table 1: Duration and Exclusion

α	$\lambda = 0.1$			$\lambda = 1$			$\lambda = 12$		
	d	$N.p$	$P\%$	d	$N.p$	$P\%$	d	$N.p$	$P\%$
0	6.7			1.0			0.1		
$\frac{1}{8}$	8.0	1.17	5.6	3.2	1.02	11.2	2.4	1.01	11.7
$\frac{1}{2}$	11.6	1.73	28.9	8.6	1.27	39.5	8.2	1.23	40.7
$\frac{9}{10}$	17.0	4.73	60.7	14.7	3.25	70.9	14.4	3.14	71.8

Table 2 gives two measures of relative values for each cell: the ratio of value to the first best (e.g. one that would be achieved with $d = 1/r$ for all innovations) and the ratio of the value obtained with no exclusions (the natural rate) to our constrained optimal value. In all cases higher recurrence rates α reduce the gap between the constrained optimal and the first best. This confirms the intuition highlighted before, that recurrence in innovation enhances incentives for investment. We also find that the gap is widened when the frequency of innovation λ increases. Our results show that the gain from exclusion – as opposed to just providing the natural rate of protection – are very moderate when the frequency of innovation λ is low but are very large for when it is high. Interestingly, this suggests that a patent system that eventually excludes competing innovators is more desirable in areas with faster rates of innovation.

The numerical results also highlight the importance of repeated innovation. The first row can be interpreted as a situation when innovators don't recur, while the $\alpha = 1/2$ case can be identified with a symmetric duopoly of innovators. Since for both cases the unconstrained optimum is the same, the values in Table 2 can be used to compare their performance. The range goes from a 40% increase in value for $\lambda = 0.1$ to an over 5 fold increase for $\lambda = 1$ or $\lambda = 12$. This shows that the concentration of innovators is particularly important in areas where there is a fast rate of innovation.

4 Complete Exclusion Rights and Competition "For the Market"

In this section we relax the requirement of only granting forward exclusion, giving the rights over the whole ladder to one firm at a time, which we

Table 2: Values

α	$\lambda = 0.1$		λ =1		λ =12	
	V/opt	V_nat/V	V/opt	V_nat/V	V/opt	V_nat/V
0	0.56	1.0	0.09	1.0	0.08	1.0
$\frac{1}{8}$	0.62	0.99	0.21	0.52	0.13	0.07
$\frac{1}{2}$	0.79	0.96	0.55	0.35	0.50	0.03
$\frac{9}{10}$	0.98	0.99	0.92	0.61	0.90	0.09

call the leading firm. As argued in the introduction, on the class of patent mechanisms this maximizes ex-ante incentives for innovation as it rewards an innovator for *all* contributions at any time where that innovator is profiting, and not just the most recent sequence of consecutive innovations as is the case with forward exclusion. In the language of [16], this corresponds to a patent with infinite lagging breadth. Competition for the market is the motivation for innovation. Because firms sell the joint output of the two firms, the opponent's innovation decisions matter for the total profits of a given innovation decision; however, since the innovations contribute to profits additively, the impact of the opponent remains separable from a given firm's profits, and we can solve the innovation problem for a given firm independently of the opponents' choice.

Given there are no static distortions, this patent mechanism maximizes total discounted welfare from innovations, as it maximizes the scarce market time delivered to the innovator for all of his innovations. As social value and the sum of private value for the two firms are identical, one can interpret this section also as a private contract, signed at time zero, between the duopolists to maximize their joint payoff. In other words, it constitutes the profit maximizing ex ante licensing agreement, where licenses cannot be made to be a function of Δ, for instance because it is unobserved or non-verifiable in court. In practice, cumulative innovations that embody many innovations are often difficult to disentangle into their contributing parts in a way that would allow rewards to be based on individual Δ. Patent pooling arrangements such as wireless network standards and the MPEG patent pool contain thousands of separate innovations accumulated into the state of the art product.

For conciseness we develop these results for the case where all ideas are identical.

4.1 Dynamic Program

In the appendix, we show that scarcity of time requires that all instants be allocated to one innovator or the other. As a result the planner's problem can be written as a function of the outstanding duration d promised to innovator 1. As the other innovator will be profiting from all innovations in the complementary time, its entitled duration is $1/r - d$. This provides a simple recursive representation with state variable d.

The outstanding duration d of firm one evolves in an optimal fashion as a function of contingencies. If innovator 1 receives the next innovation we denote the new promised duration by d_1; if innovator 2 is the first to innovate, the new duration for innovator 1 will be denoted by d_2. Innovator two receives durations $\frac{1}{r} - d_1$ and $\frac{1}{r} - d_2$, respectively. While $0 < d_1 < \frac{1}{r}$, all ideas will be implemented, notwithstanding the fact that incentives for investment will be low for one player when the promised duration is close to one of these extremes.

The planner must also decide whom to allocate rights to sell in the interim period before the next arrival; in the prior section it was always to the most recent innovator. Here $x \in [0, 1]$ determines the share of instants until the next arrival where innovator 1 has exclusive rights. So when $x = 1$, innovator one is the seller while for $x = 0$ it is innovator two. The dynamic program that solves the optimal assignments is given by the following functional equation:

$$
\begin{aligned}
rV(d) &= \max_{d_1,d_2,x} \left\{ \begin{array}{c} \lambda\left((R(d_1) + V(d_1) - V(d)\right) + \\ \lambda\left(R(\frac{1}{r} - d_2) + V(d_2(\theta)) - V(d)\right) \end{array} \right\} \qquad (5) \\
& \quad s.t. \\
rd &= x + \lambda(d_1 - d) + \lambda(d_2 - d)
\end{aligned}
$$

Since the problem is symmetric, the value function must be symmetric around the midpoint $\frac{1}{2r}$,so we restrict our discussion on the shape of V to the set $[1/2r, 1/r]$. We first study when the promise keeping constraint binds, which gives some basic insight into the shape of V. This question is analogous to the question of when x is strictly between zero and one, since examining the above problem it is clear that x could not be interior unless the promise keeping constraint were not binding.

As in the case of forward exclusion, we can identify a natural rate corresponding to the case where selling rights are assigned to the most recent innovator rights until the next arrival. This corresponds to the duration $d = 1/(r + \lambda)$ from the last section, when there is no exclusion and therefore all innovations are treated identically. Denote by $\hat{d}$ the duration this offers to the most recent arrival. By symmetry, when the other firm has an arrival duration drops to $1/r - \hat{d}$. Therefore $\hat{d}$ solves

$$r\hat{d} = 1 + \lambda(1/r - \hat{d} - \hat{d})$$

or $\hat{d} = \frac{r+\lambda}{r(r+2\lambda)} > 1/2r$. The planner can offer every arrival this duration. If the planner offers any innovator more for one of its arrivals, however, it will curtail the ability to reward the other innovator in the future. We study this trade off using the dynamic program in the remainder of this section.

4.2 Characterization

The appendix shows that V is once again concave; since it is symmetric, it is maximized at $1/2r$. This is intuitive: when duration promise is identical for the two agents, they will be treated identically upon the next arrival, setting $d_1 = 1/r - d_2$ which is best since R is concave. Note that having the agents treated identically requires

$$\begin{aligned} rd &= x + \lambda(1/r - d_2 - d) + \lambda(d_2 - d) \\ x &= (r + 2\lambda)d - \lambda/r \end{aligned}$$

An identical allocation to future innovations is feasible if $d \in [1/r - \hat{d}, \hat{d}]$, by adjusting x between zero and one. Intuitively, in this case, the planner can deliver any asymmetric preference by using x, leaving the balance of the duration promise identical across agents when the next innovation arrives, and allowing $d_1 = 1/r - d_2$. As a result it is immediate that

Lemma 10. *$V(d)$ is constant in the range of* $[1/r - \hat{d}, \hat{d}]$

This range is the one where the promise keeping constraint does not bind.[17] Clearly, outside of this range the planner can no longer have $d_1 = 1/r - d_2$, and therefore value must be lower, since concavity in R dictates losses when the next arrivals are treated differently. It is clear that it is never optimal to choose a point in the interior of the flat portion, since raising the current innovator's promise has no cost. The following lemma shows that the planner must go even further.[18]

[17] It is easy to show that this is a self generating property of the value function, and therefore must be true of $V(d)$ which is a fixed point of the Bellman operator.

[18] Proofs of results from this point forward are contained in the Appendix.

Proposition 11. $d_1(d) > d$

Since $d_1(d)$ cannot depend on d for $d \in [1/r - \hat{d}, \hat{d}]$, the proposition implies that $d_1(d) > \hat{d}$ for d in that range. For duration promises in excess of $\hat{d}$, the first order condition for d_1 and concavity of V shows that duration is an increasing sequence for any consecutive ideas by innovator 1, just as in the prior case. An increasing sequence on an interval must converge, and by the first order condition for d_1 it cannot converge to $d < 1/r$, where $R' > 0$. Therefore, sequences of arrivals by firm one get arbitrarily close to a promised duration of $1/r$. Duration rises and falls with arrivals by the two firms; the two firms engage in a "tug of war" for duration.

An interesting feature is the evolution of x during this period. Suppose that, due to past success by innovator 1, $d > \hat{d}$. Then $x = 1$: the intervening period until the next innovation is entirely allocated to firm 1. If d is high enough, it may be the case that $d_2 > \hat{d}$, meaning that even after an innovation by firm 2, firm one will still maintain rights to all innovations, including the new one that firm 2 has just produced. Firm 2 only receives rights when a sufficient number of innovations by it have moved duration below $1/r - \hat{d}$, i.e. past the decreasing portion of the value function. This conforms to the idea that trailing firms need to make sufficient progress before their innovations are deemed to "not infringe" on the current leader's patent. Here, during the period of infringement, the leader maintains rights to all innovations, including the ones being invented by the laggard firm, as if it has the ability to costlessly license the infringing ideas. The payoff to the trailing firm is the eventual ability to sell a product that embodies the entire history of ideas, once their duration promise is sufficiently high.

An alternative interpretation of the optimal allocation is not as patent policy alone, but as favorable treatment from a regulator more generally. Suppose favorable treatment allows the firm to reap all the benefits of innovations from any firm, for instance by the incumbent firm negotiating licensing contracts that extract full surplus. Here the optimal policy uses such favorable treatment as an incentive device. The regulator favors the leader until the laggard has had sufficient innovations to be the new leader, at which time the regulator shifts its favorable treatment to that firm. Laggard innovators innovate for eventual favorable treatment. In this sense the optimal allocation has a strong sense that it entails competition for the market as an incentive device; this competition for the market, however, eventually leads to near permanent monopolization by some firm.

Table 3: Values: Total exclusion vs. Forward exclusion

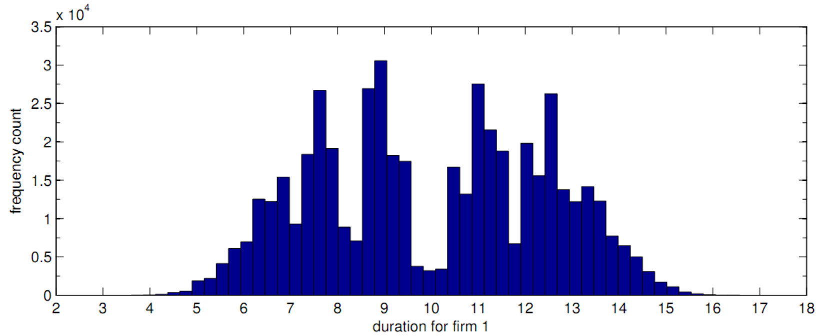
2λ	Forward (V/opt)	Total (V/opt)	Ratio
0.1	0.79	0.90	1.13
1	0.55	0.80	1.46
12	0.50	0.76	1.51

The contract can equally be viewed as an ex ante licensing contract signed between the two firms. An interesting feature is that the optimal contract at no point in time shares the profits generated by the joint research, in the sense of splitting the profits between the firms; the licensing always takes the form of dynamic splitting, where one firm is rewarded for their work by having a longer time during which they profit completely. The contract relies on being able to pre-specify the extreme rights (x being either zero or one) as a function of history that boils down to the state d. The policy is a sort of duopoly patent pool, where the firms pool their patents and profits flow to one of the pool members based on their relative contribution, measured through d.

To illustrate the potential gains and the degrees of asymmetry that can arise, we provide results of numerical computations. The benchmark values used are $\lambda = 1/2$, $r = 0.05$ and a quadratic cost function. Table 3 give the ratio of values to the unconstrained optimal achieved through forward exclusion and total exclusion. Since the first best value is the same in both scenarios, the ratio given in the third column shows the relative advantage of total exclusion, ranging from about 10 to 50%. Gains are larger again in industries with higher pace of innovation, due mostly to the shortcomings of protection under forward exclusion.

Figure 3 gives the distribution of duration generated by a sequence of 500,000 draws and iterative application of the policy function. For sufficiently many draws the range of observed values in would go from zero to 20=1/.05, corresponding to the absorbing extreme points of duration. In spite of the fact that these two points are absorbing, convergence is so slow and unlikely that for a finite sample of as large a size as the one we have generated, there appears to be a non-degenerate distribution. This distribution shows the prevalence of asymmetry (meaning durations that exceed the midpoint 10), obviously in favor of both players. The average duration to the right side of the midpoint is slightly over 12 (where player one is assigned

Figure 3: Distribution of duration



50% more duration than player two) and symmetrically below 10 is approximately 8 (where player two is assigned 50% more duration than player one.) Durations of 15 are clearly in the support (here player one is assigned 3 times the duration of player two.). As a reference point, it is worth noting that the natural duration $\hat{d}$ (the one that provides *equal treatment* to all innovations) is roughly 10.5, considerably away from the area where the mass of the distribution is concentrated. *Unequal treatment* seems more the norm resulting from the optimal incentive mechanism.

If one accounts for static distortions, so that that $R(d)$ is not maximized at $1/r$ but rather at some static welfare maximizing patent as in [3] and [15], would serve to reduce the duration promises, but not change the qualitative results.

5 Conclusions

We have characterized the solution to the problem facing a planner who must allocate rights to production across two firms who can use those rights to make profits, and in turn are encouraged to innovate by the provision of the rights. It allows us to address the question of what distribution of rights arises from planner's solution, and in particular how much the market becomes "concentrated." The planner, because he can allocate rights to a single firm for multiple innovations at any point in time, backloads rewards, giving the firm with the preponderance of the future promises an exclusive right to all of the current profits. The optimal policy we study leads to near-permanent monopoly, in the sense that one firm is excluded even though it is getting useful ideas. We show that these basic results hold even if the planner is forced to use a restricted set of polices where rights are is always granted immediately for any innovation that is implemented (forward exclusions). In that case, the optimal allocation can be decentralized through a simple set of patent fees: one for a patent with no forward breadth, and an additional fee that gives the innovator infinite future breadth. One can interpret the results as casting light on regulatory policies designed to foster competition "for the market." When the state dependence of rights is combined with a dynamic model of competition for the market, competition dies out in the long run.

Appendix A: Proofs for Forward Exclusion Rights

Lemma 12. *$V(d)$ is concave and differentiable on $[1/(r+\lambda), 1/r]$.*

Proof of Lemma 12

It is convenient to rewrite the value function as follows:

$$\begin{aligned} V(d) &= \max_{d_1(\theta), d_2(\theta), \bar{\theta}} \beta \int \left[R(d_1(\theta), \theta) + V(d_1(\theta)) \right] dF \\ &\quad + \beta \int_{\bar{\theta}} \left[R(d_2(\theta), \theta) + V(d_2(\theta)) \right] dF + \beta F\left(\bar{\theta}\right) V(d) \end{aligned}$$

where $\beta = \lambda / (r + 2\lambda)$. Likewise, promise-keeping can be rewritten as:

$$d = \beta \left\{ \frac{1}{\lambda} + \int d_1(\theta)\, dF(\theta) + F\left(\bar{\theta}\right) d \right\}$$

For the inductive hypothesis, assume the right hand side value function is concave. Take two values d and d' and for arbitrary $\gamma \in (0,1)$ let $d^\gamma = \gamma d + (1-\gamma) d'$. Let $(d_1(\theta), d_2(\theta)), \bar{\theta}$ and $\left(d_1'(\theta), d_2'(\theta), \bar{\theta}'\right)$ denote the optimal policies at d and d' Assume without loss of generality that $\bar{\theta}' \geq \bar{\theta}$.

We now describe a feasible policy for d^γ that does at least as well as $\gamma V(d) + (1-\gamma) V(d')$. Let $d_1^\gamma(\theta) = \gamma d_1(\theta) + (1-\gamma) d_1'(\theta)$. Define $\bar{\theta}^\gamma$ by

$$\frac{F\left(\bar{\theta}'\right) - F\left(\bar{\theta}^\gamma\right)}{F\left(\bar{\theta}'\right) - F\left(\bar{\theta}\right)} = \gamma.$$

Note that it must be that $\bar{\theta}^{\gamma} \in [\bar{\theta}, \bar{\theta}']$. For $\theta > \bar{\theta}^{\gamma}$ let

$$d_2^{\gamma}(\theta) = \begin{cases} d_2(\theta) & \theta < \bar{\theta}' \\ \gamma d_2(\theta) + (1-\gamma) d_2'(\theta) & \theta \geq \bar{\theta}' \end{cases}$$

Finally, when an outsider arrives and is not implemented, i.e. $\theta < \bar{\theta}^{\gamma}$, let the duration promise of the incumbent be d^{γ}if $\theta < \bar{\theta}$ and d' if $\theta \in [\bar{\theta}, \bar{\theta}^{\gamma}]$

We first check that the promise keeping constraint is still satisfied. The delivered duration is

$$\beta \left\{ \frac{1}{\lambda} + \int d_1^{\gamma}(\theta)\, dF(\theta) + F(\bar{\theta})\, d^{\gamma} + \left(F(\bar{\theta}^{\gamma}) - F(\bar{\theta})\right) d' \right\}$$

where the last two terms reflect the two possible continuation durations, d^{γ} and d', that can result following an arrival by the outsider that is not implemented. Note that those last term can be written as

$$\left(F(\bar{\theta}^{\gamma}) - F(\bar{\theta})\right) \frac{F(\bar{\theta}') - F(\bar{\theta})}{F(\bar{\theta}') - F(\bar{\theta})} d'$$

which is just $\left(F(\bar{\theta}') - F(\bar{\theta})\right)(1-\gamma)d'$. This implies promise keeping is satisfied at d^{γ}, since it is exactly satisfied for $\theta < \bar{\theta}$ and $\theta > \bar{\theta}'$ by delivering the convex combination of the two policies pointwise, and on $\theta \in [\bar{\theta}, \bar{\theta}']$, where the policies for θ and θ' deliver 0 and d', respectively, to the incumbent, the modified policy gives exactly $(1-\gamma)d'$.

For arrivals by the incumbent and arrivals by the outsider with $\theta < \bar{\theta}$ and $\theta > \bar{\theta}'$ the policy gives higher value immediately by the inductive assumption. For $\theta_2 \in \left[\bar{\theta}, \bar{\theta}'\right]$ the associated continuation value for the planner is

$$\begin{aligned} & \int_{\bar{\theta}^{\gamma}}^{\bar{\theta}'} \left[R(d_2(\theta), \theta) + V(d_2(\theta))\right] dF + \left(F(\bar{\theta}^{\gamma}) - F(\bar{\theta})\right) V(d') \\ \geq\ & \tfrac{F(\bar{\theta}') - F(\bar{\theta}^{\gamma})}{F(\bar{\theta}') - F(\bar{\theta})} \int_{\bar{\theta}}^{\bar{\theta}'} \left[R(d_2(\theta), \theta) + V(d_2(\theta))\right] dF + \left(F(\bar{\theta}^{\gamma}) - F(\bar{\theta})\right) V(d') \\ =\ & \gamma \int_{\bar{\theta}}^{\bar{\theta}'} \gamma \left[R(d_2(\theta), \theta) + V(d_2(\theta))\right] dF + \left(F(\bar{\theta}') - F(\bar{\theta})\right)(1-\gamma) V(d') \end{aligned}$$

where the inequality results from the fact that $R(d_2(\theta), \theta) + V(d_2(\theta))$ is increasing in θ, which follows from optimality.[19] Note that the last line

[19]Given concavity of the RHS, one can apply the proof that is provided below.

is exactly the convex combination of what the two policies deliver on this interval.

To show that V is differentiable, for any d, define $\underline{V}(\epsilon)$ by

$$r\underline{V}(\epsilon) = \left\{ \begin{array}{c} \lambda \left(\int (R(d_1(\theta), \theta) + V(d_1(\theta)) - V(d)\right) dF(\theta) + \\ \lambda \int_{\bar{\theta}_\epsilon}^{\infty} (R(d_2(\theta), \theta) + V(d_2(\theta)) - V(d))\, dF(\theta) \end{array} \right\}$$

Where, for duration d, $\bar{\theta}_\epsilon$ solves

$$r(d+\epsilon) = 1 + \lambda(\int d_1(\theta) dF(\theta) - (d+\epsilon)) - \lambda(1 - F(\bar{\theta}_\epsilon))(d+\epsilon)$$

so that promise keeping is satisfied for fixed $d_1(\theta)$ and $d_2(\theta)$, which are taken to be their optimum values at duration d.[20] The function $\underline{V}$ is differentiable, $\underline{V}(0) = V(d)$, and, since it is computed for a feasible policy, $\underline{V}(\epsilon) \leq V(d+\epsilon)$, so V is differentiable.

Proof of Proposition 3:

Proof. Starting at d, let $\{d_n, p_n\}$ denote the sequence giving the continued duration and probability of replacement after n consecutive arrivals of he incumbent. Naturally, promise keeping satisfies:

$$rd_n = 1 + \lambda (d_{n+1} - d_n) - \lambda p_n d_n.$$

We now show that in the optimal mechanism, $p_{n+1} > 0$ implies $p_n = 1$. Suppose towards a contradiction that $p_{n+1} > 0$ and $p_n < 1$. Consider an alternative path identical to the original one with the exception that $\tilde{p}_n = p_n + \varepsilon$ and $\tilde{p}_{n+1} = p_{n+1} - \delta$ where δ is calculated so that $\tilde{d}_n = d_n$. We show now that there exist such $\delta < p_{n+1}$ for ε small. Using the functional equation for d_n on the original and alternative path:

$$\begin{aligned} rd_n &= 1 + \lambda (d_{n+1} - d_n) - \lambda p_n d_n \\ &= 1 + \lambda \left(\tilde{d}_{n+1} - d_n\right) - \lambda \tilde{p}_n d_n \end{aligned}$$

giving the necessary and sufficient condition:

$$\tilde{d}_{n+1} - d_{n+1} = (\tilde{p}_n - p_n)\, d_n \equiv \varepsilon d_n.$$

[20] If necessary, $d_2(\theta)$ is extended in any continuous way below $\bar{\theta}_0 = \bar{\theta}$

Now observe that since $p_{n+1} > 0$ it follows that $\frac{1}{r+2\lambda} \leq d_{n+1} < \frac{1}{r}$ so it can be increased by reducing p_{n+1}. This proves that the alternative path is feasible and gives $d_n = \tilde{d}_n$ for $\varepsilon > 0$ small enough. □

Finally, we show that this alternative path is strictly better. Both sequences of durations are identical except for $\tilde{d}_{n+1} > d_{n+1}$. From the perspective of the incumbent, it induces the same sequence of investments except for $\tilde{\Delta}_{n+1} > \Delta_{n+1}$, so it is strictly better. Moreover, the variation is neutral with respect to the expected value generated when replacing the incumbent since since initial duration (the expected discounted time of this arrival) is unchanged.

Having established that $p_{n+1} > 0$ implies $p_n = 1$, it follows that the sequence $\{p_n\}$ is weakly decreasing. Moreover, there can be at most one period where $0 < p_n < 1$ for otherwise the condition that we proved would be violated. This proves that the optimal sequence has $p_n = 1$ for $n = 1, .., N-1$, $0 \leq p_N < 0$ and $p_n = 0$ for $n \geq N+1$. This also implies that the sequence d_n is increasing to the point where $d_{N+1} = \frac{1}{r}$ and the outsider is excluded forever.

Proof of Lemma 4:

Proof. Starting at time $t = 0$ consider the stopping time T_1 defined by player one gets its first arrival before player two. This stopping time has an associated density $f(t) = \lambda \int e^{-2\lambda t} dt$ and associated expected discount factor $Ee^{-rT_1} = \lambda \int e^{-(r+2\lambda)t} dt = \frac{\lambda}{r+2\lambda}$. Denote this expected discount factor by β. By independence, the stopping time T_{1n} defined by player one gets its n^{th} arrival prior to player two getting an arrival has expected discount factor β^n. This is also the expected discount factor for the event: "player two gets its first arrival after $n-1$ arrivals of player one." Start with the baseline policy where all arrivals are implemented granting the incumbent player (say player one) duration $d^* = \frac{1}{r+\lambda}$. Consider the following deviation: if after $n-1$ consecutive arrivals for player one there is an arrival for player two, do not implement that arrival but then return to the baseline plan. If instead player one gets its n^{th} arrival, return to the baseline plan. This is a plan that delivers a sequence $\{d_0, d_1, d_2, ..., d_{n-1}, d_{n+1,...}\}$ with $d_i > d^*$ for all $i < n$ and equal to d^* for $i \geq n$. The increased durations for player one have the benefit of larger innovations but there is the cost of missing one potential innovation and the corresponding value $R^* = R(d^*)$.

As of time zero, the expected discounted cost is $\beta^n R^*$. The calculation of the benefits is slightly more complex and follows here. We start with $n-1$ and using the recursive definition of duration, it easily follows that:

$$(r+2\lambda)(d_{n-1}-d^*) = \lambda d^*$$

Incidentally, note that

$$\frac{d_{n-1}-d^*}{d^*} = \frac{\lambda}{r+2\lambda} = \beta.$$

More generally,

$$(r+2\lambda)(d_i-d^*) = \lambda(d_{i+1}-d^*)$$

which dividing through by d^* gives

$$\frac{d_i-d^*}{d^*} = \beta\frac{d_{i+1}-d^*}{d^*} = \beta^{n-i}.$$

In consequence, $d_i = (1+\beta^{n-i})\,d^*$ and in particular $d_0 = (1+\beta^n)\,d^*$. The value of the given alternative plan:

$$\begin{aligned} W_n - W^* &= \sum_{i=0}^{n-1}\beta^i\left(R(d_i)-R(d^*)\right)-\beta^n R(d^*) \\ &\geq \sum_{i=0}^{n-1}\beta^i R'(d_{n-1})(d_i-d^*)-\beta^n R(d^*) \\ &= \sum_{i=0}^{n-1}\beta^i R'((1+\beta)\,d^*)\,\beta^{n-i}d^*-\beta^n R(d^*) \\ &= \beta^n\left[nR'((1+\beta)\,d^*)\,d^*-R(d^*)\right] \end{aligned}$$

It is straightforward to see that since $(1+\beta)\,d^* < \frac{1}{r}$, $R'((1+\beta)\,d^*) > 0$ and so for sufficiently large n, $W_n > W^*$. □

Derivation of formula for $V'(d)$

In region with $p = 1$

Using the functional equations:

$$\begin{aligned} (r+\lambda)\,V(d) &= \lambda\alpha\left[R(d_1)+V(d_1)\right]+\lambda(1-\alpha)\left[R(d_2)+V(d_2)\right] \\ (r+\lambda)\,d &= 1+\lambda\alpha d_1 \end{aligned}$$

Using the second equation, $\lambda\alpha\partial d_1/\partial d = r + \lambda$ and now totally differentiating the first equation with respect to d after substitution yields:

$$V'(d;\alpha) = R'(d_1) + V'(d_1;\alpha). \tag{6}$$

In region with $0 < p < 1$

The promise-keeping constraint can be used to rewrite the functional equation in a more convenient way:

Multiplying both sides of the PK constraint by r and adding to both sides $\lambda(1-\alpha)p$, gives:

$$[r + \lambda(\alpha + (1-\alpha)p)]\, rd + \lambda(1-\alpha)p = r + \lambda(\alpha + (1-\alpha)p)$$

so

$$\frac{\lambda(1-\alpha)p}{r + \lambda(\alpha + (1-\alpha)p)} = 1 - rd. \tag{7}$$

In addition, solving for rd and simplifying gives

$$rd = \frac{r + \lambda\alpha}{r + \lambda(\alpha + (1-\alpha)p)}$$

implying

$$\frac{\lambda\alpha}{r + \lambda(\alpha + (1-\alpha)p)} = \frac{\lambda\alpha}{r + \lambda\alpha} rd. \tag{8}$$

Now substituting (7) and (8) in the dynamic programing equation gives:

$$V(d) = \frac{rd}{r + \lambda\alpha}\lambda\alpha\left(R\left(\frac{1}{r}\right) + V\left(\frac{1}{r}\right)\right) + (1 - rd)(R(d_2) + V(d_2)) \tag{9}$$

Using $V\left(\frac{1}{r}\right) = \frac{\lambda\alpha}{r} R\left(\frac{1}{r}\right)$ gives:

$$R\left(\frac{1}{r}\right) + V\left(\frac{1}{r}\right) = \frac{r + \lambda\alpha}{r} R\left(\frac{1}{r}\right) = \frac{r + \lambda\alpha}{\lambda\alpha} V\left(\frac{1}{r}\right)$$

and substituting this in (9) gives the simplified dynamic programming equation:

$$V(d) = rdV\left(\frac{1}{r}\right) + (1 - rd)(R(d_2) + V(d_2)) \tag{10}$$

Solving forwards

Equations (6) and (10) can be used to forward substitute $V'(d_n)$ and this gives:

$$V'(d;\alpha) = \sum_{n=1}^{N} R'(d_n) - r\left[R(d_2) + V(d_2) - V\left(\frac{1}{r}\right)\right]. \tag{11}$$

Proof of Proposition 6

Denote by W_N the value of an outside firm, upon receiving an idea and paying f, if he excludes after N arrivals. We set $f = W_N$. This implies that the value of an outsider is zero. We need to show that, first, we can set t so that it is optimal to pay t after N arrivals. Then it is immediate that the outsider without an idea does not find it worthwhile to pay f as one who does is indifferent between paying f or not.

Consider the deviation of paying t (the foreclosure fee), n arrivals after the initial payment of f. The value of this plan is denoted W_n for arbitrary n . For any such deviation strategy we have associated durations when n steps from foreclosure. They can be solved recursively from

$$rd_n = 1 - \lambda d_n + \lambda(d_{n-1} - d_n)$$

with $d_0 = \frac{1}{r}$. Denote $\beta = \lambda \int e^{-(r+2\lambda)t}dt = \frac{\lambda}{r+2\lambda}$. The recursion implies

$$d_n = \frac{(1-\beta^n)}{1-\beta} + \frac{\beta^n}{r}.$$

We can divide up rewards into profits and expected payment of fees.

$$W_n = v_n - \beta^n t$$

The profits from selling follow a simple recursion:

$$v_{n+1} = \pi_{n+1} + \beta v_n$$

where $\pi_n = max_x d_n x - c(x)$. This recursion decomposes the reward from innovating into two parts. First, there is the expected profits from selling the current increment. Since that innovation lasts for d_n units of time, its profits are π_n. If the incumbent gets the next idea (embodied in the discounting by

β), they will face the same problem as any innovator $n-1$ steps from foreclosure, except for the profits they make from the increment they generated from earlier innovations.

We want to show that an arbitrary n (in particular N) can be made the maximum of W_n by appropriate choice of t. We start by showing that we can make it a local maximum, i.e.

$$\begin{aligned} v_n - \beta^n t &\geq v_{n+1} - \beta^{n+1} t \\ v_n - \beta^n t &\geq v_{n-1} - \beta^{n-1} t \end{aligned}$$

Note that the first is equivalent to

$$\beta^n (1-\beta) t \leq (1-\beta) v_n - \pi_{n+1}$$

and the second is equivalent to

$$\beta^n (1-\beta) t \geq \beta((1-\beta) v_{n-1} - \pi_n)$$

Note that $v_n > \beta v_{n-1}$ and $\pi_n > \pi_{n+1}$, so these two can always be satisfied simultaneously for appropriate t.

The next two claims verify that any local maximum is also a global one. This completes the proof.

Claim 13. If $v_n - \beta^n t \geq v_{n-1} - \beta^{n-1} t$ then $v_{n-1} - \beta^{n-1} t \geq v_{n-2} - \beta^{n-2} t$.

Proof. Rewrite the first inequality as $\pi_n + \beta v_{n-1} - \beta^n t \geq \pi_{n-1} + \beta v_{n-2} - \beta^{n-1} t$. Observing that $\pi_n < \pi_{n-1}$ it follows that $\beta v_{n-1} - \beta^n t \geq \beta v_{n-2} - \beta^{n-1} t$. Dividing through by β we get $v_{n-1} - \beta^{n-1} t \geq v_{n-2} - \beta^{n-2} t$. □

Claim 14. $v_n - \beta^n t \geq v_{n+1} - \beta^{n+1} t$ implies $v_{n+1} - \beta^{n+1} t \geq v_{n+2} - \beta^{n+2} t$.

Proof. Multiply the first inequality by β and substituting on the left hand side βv_n by $v_{n+1} - \pi_{n+1}$ and on the right hand size βv_{n+1} by $v_{n+2} - \pi_{n+2}$ gives:

$$v_{n+1} - \pi_{n+1} - \beta^{n+1} t \geq v_{n+2} - \pi_{n+2} - \beta^{n+2} t.$$

Observing that $\pi_{n+1} > \pi_{n+2}$ this implies that $v_{n+1} - \beta^{n+1} t \geq v_{n+2} - \beta^{n+2} t$. □

Proof of Propositions 7 and 8

Consider the problem:

$$\begin{aligned} rV(d;\alpha) &= max_{d_1,d_2,p}\alpha\lambda\left[R(d_1)+V(d_1;\alpha)\right]+(1-\alpha)\lambda p\left[R(d_2)+V(d_2;\alpha)\right] \\ s.t\ d &= 1+\alpha\lambda(d_1-d)-(1-\alpha)pd \end{aligned}$$

We show that $V(d;\alpha)$ is increasing in α for $d > d_2(\alpha)$ and small change in α.For induction, assume right hand side V is increasing in α. Take $\alpha' > \alpha$. Consider the following (suboptimal policy). For an arrival of the outsider, choose the same p and d_2 that are optimal for α. For an arrival of the incumbent grant the incumbent the same value d_2 with probability $p\left(\frac{\alpha'-\alpha}{\alpha'}\right)$ and d_1' with probability α/α' and the same d with probability $\frac{1-p}{\alpha'}(\alpha'-\alpha)$. Upon substitution, the promise-keeping constraint reads

$$rd = 1+\lambda\alpha(d_1'-d)+p\lambda(\alpha'-\alpha)d_2-\lambda(1-\alpha)pd,$$

identical to the promise-keeping for α except for the third term. It follows that $d_1' < d$. Choose α' sufficiently close to α so that $d_1' > d_2$. Denoting by $\tilde{V}$ the value of this policy, it satisfies the functional equation:

$$\begin{aligned} r\tilde{V} &= \lambda\alpha\left(R(d_1')+V(d_1',\alpha')-\tilde{V}\right)+\lambda(1-p)(\alpha'-\alpha)\left(R(d)+V(d;\alpha)-\tilde{V}\right) \\ &\quad +\lambda(1-\alpha)p\left(R(d_2)+V(d_2;\alpha')-\tilde{V}\right) \\ &\geq \lambda\alpha\left(R(d_1')+V(d_1',\alpha)-\tilde{V}\right)+\lambda(1-\alpha)p\left(R(d_2)+V(d_2;\alpha)-\tilde{V}\right) \\ &> \lambda\alpha\left(R(d_1)+V(d_1;\alpha)-\tilde{V}\right)+\lambda(1-\alpha)p\left(R(d_2)+V(d_2;\alpha)-\tilde{V}\right) \end{aligned}$$

where the first inequality follows from the induction hypothesis and the fact that since this policy is suboptimal $V(d;\alpha) \geq \tilde{V}$ and the second inequality from the fact that strict concavity$R(\cdot)+V(\cdot;\alpha)$ implies it is strictly decreasing for durations above its maximizer d_2 and that $d < d_1' < d_1$. This proves that $\tilde{V} > V(d,\alpha)$. Since the policy considered is feasible (and actually suboptimal), it follows that $V(d,\alpha') > V(d,\alpha)$.

We consider now the proof of Proposition 8. Since the value functions are concave, they are almost everywhere differentiable. We show that $\partial V(d;\alpha)/\partial d$ is increasing in α. Using the promise-keeping constraint and maintaining fixed p (an envelope condition argument)

$$\partial d_1/\partial d = \frac{r+\lambda(\alpha+(1-\alpha)p)}{\lambda\alpha}.$$

Differentiating the functional equation it follows that:

$$(r+\lambda(\alpha+(1-\alpha p)))(\partial V(d;\alpha)/\partial d)=\lambda\alpha\frac{\partial}{\partial d_1}(R(d_1)+V(d_1;\alpha))\frac{\partial d_1}{\partial d}$$

that after substitution implies $\partial V(d;\alpha)/\partial d=\frac{\partial}{\partial d_1}(R(d_1)+V(d_1;\alpha))$. Assume inductively that $\partial V(d_1;\alpha)$ is increasing in α. Using the promise keeping condition, d_1 decreases in α and by the strict concavity of R and the induction assumption it follows that $\partial V(d;\alpha)/\partial d$ is increasing in α.

Appendix B: Optimality of Complete Exclusion Rights

We begin with the most general problem, where both firms have promises that may not add up to all available time, and prove that the optimal policy can be solved by the dynamic program (5).

We introduce the following notation. If an innovation by firm 1 arrives, the planner offers preferential treatment for that new innovation for duration d_1^n. It continues preferential treatment for the innovator's previous innovation (or innovations), which are owed d, for duration d_1^c. The planner will then enter the next instant with promise equal to the maximum of d_1^n and d_1^c, since the outstanding duration that cannot be allocated to other firms is the larger of those promises. We will argue below that optimally $d_1^n=d_1^c$, and therefore we will eventually just use d_1 to denote the new promise. If innovator two has the next idea, then innovator 1's duration becomes d_2. We keep track of the duration promise to the two firms by d and $\underline{d}$; we show below that it is sufficient to track only one. In the interim, we speak generically about duration as d; everything is symmetric across the innovators, so all statements apply equally to $\underline{d}$. In order to make everything completely symmetric, we refer to the promise to firm two in the event that firm one arrives by $\underline{d}_2$, and so on. If nothing arrives, the planner may change the duration promise by

$\dot{d}$. The dynamic program is

$$rV(d,\underline{d}) = \max_{\substack{d_1^n,d_1^c,d_2,\dot{d},x \\ \underline{d}_1^c,\underline{d}_1^n,\underline{d}_2,\dot{\underline{d}},\underline{x}}} \left\{ \begin{array}{c} \lambda\left(R(d_1^n) + V(\max\{d_1^n, d_1^c\}, \underline{d}_2) - V(d,\underline{d})\right) + \\ \lambda\left(R(\underline{d}_1^n) + V(d_2, \max\{\underline{d}_1^n, \underline{d}_1^c\}) - V(d,\underline{d})\right) + \\ V_1(d,\underline{d})\dot{d} + V_2(d,\underline{d})\dot{\underline{d}} \end{array} \right\} \quad (12)$$

$$s.t. \quad (13)$$

$$rd = x + \lambda(d_1^c - d) + \lambda(d_2 - d) + \dot{d} \quad (14)$$

$$r\underline{d} = \underline{x} + \lambda(\underline{d}_1^c - \underline{d}) + \lambda(\underline{d}_2 - \underline{d}) + \dot{\underline{d}} \quad (15)$$

The first line of the maximand is the case where the current innovator, promised d for prior innovations, arrives with a new idea. The second line is the case where the competitor arrives with an idea. The final line is when nothing arrives. There are also the domain constraints:

$$\begin{aligned} 0 &\leq \max\{d_1^n, d_1^c\} + \underline{d}_2 \leq 1/r \\ 0 &\leq d_2 + \max\{\underline{d}_1^n, \underline{d}_1^c\} \leq 1/r \\ 0 &\leq x + \underline{x} \leq 1 \end{aligned}$$

Since greater d only makes the feasible set of possible choices of d_1 and d_2 smaller, it is immediate that $V(d,\underline{d})$ is weakly decreasing in each argument. This in turn implies that d_1^c can always be taken to be at least as big as d_1^n; if d_1^c were less, raising it and offsetting the increase by lowering $\dot{d}$ to maintain promise keeping always does at least as well, and strictly better if V is strictly decreasing. Similarly, for $d_1^c > d_1^n$, reducing d_1^c at the margin is identical to increasing $\dot{d}$, and therefore we can let $d_1^c = d_1^n \equiv d_1$. However, in the modified program where $d_1^c = d_1^n \equiv d_1$ the envelope condition is[21]

$$V_1(d,\underline{d}) + \frac{1}{r+2\lambda} V_{11}(d,\underline{d})\dot{d} = \mu(d,\underline{d})$$

where $\mu(d)$ is the Lagrange multiplier on the PK constraint for d. This coincides with the first order condition for $\dot{d}$

$$V_1(d,\underline{d}) = \mu(d,\underline{d})$$

when $\dot{d} = 0$. We therefore have the following lemma.

Lemma 15. *Suppose V is concave. Then $d_1^c = d_1^n$ and $\dot{d} = 0$*

[21] Subscripts denote derivatives.

We now verify that V is in fact concave. If it is, then imposing the earlier results we have a simplified problem[22]

$$\begin{aligned} rV(d,\underline{d}) &= \max_{\substack{d_1,d_2,x \\ \underline{d}_1,\underline{d}_2,\underline{x}}} \left\{ \begin{array}{c} \lambda\left(R(d_1)+V(d_1,\underline{d}_2)-V(d,\underline{d})\right)+ \\ \lambda\left(R(\underline{d}_1)+V(d_2,\underline{d}_1)-V(d,\underline{d})\right)+ \end{array} \right\} \qquad (16) \\ & s.t. \\ rd &= x+\lambda(d_1-d)+\lambda(d_2-d) \\ r\underline{d} &= \underline{x}+\lambda(\underline{d}_1-\underline{d})+\lambda(\underline{d}_2-\underline{d}) \qquad (17) \end{aligned}$$

Lemma 16. *V is concave*

Proof. The Bellman equation can be rewritten as

$$V(d,\bar{d}) = \frac{1}{r}\frac{\lambda}{r+2\lambda}\max(R(d_1)+R(\underline{d}_1)+V(d_1,\underline{d}_2)+V(d_2,\underline{d}_1))$$

From this we can see immediately that the Bellman operator maps concave functions into concave functions, since the convex combination of choices for two states $(d,\bar{d})$ is feasible at the convex combination of the states, and delivers more when V on the right is concave. □

Next, we make the final step in simplifying the problem. We argue that for any value of the state $(d,\bar{d})$, it must be the case that $d+\bar{d}=1/r$. Intuitively, if there were only one innovation, the planner would like to offer it $1/r$; as a result, given the many ideas that will arrive, the planner never "wastes" any instants.

Lemma 17. $d+\bar{d}=1/r$

Proof. Suppose $d+\bar{d}<1/r$. Since both d and $\bar{d}$ cannot be greater than 1, It must be the case that $x+\bar{x}=1$, since, if either duration is less than 1 the corresponding x should be increased. Since this applies at all instants, it must always be the case that $x+\bar{x}=1$ and as a result all instants are promised to one of the two innovators, that is, $d+\bar{d}=1/r$ □

[22] In the language of the dynamic program (16), the forward exclusion case adds the restriction that it $\underline{d}_1>0$, then $d_2=0$, and if $\underline{d}_2>0$ then $d_1=0$. One can use this structure to derive the problem studied in that section.

Proof of Proposition 11:

Proof. Since it is clear that $d_1(\hat{d})$ can never be less than $\hat{d}$, we focus on the case where $d_1(\hat{d}) = \hat{d}$. Since promise keeping does not bind, this implies that $d_2(\hat{d}) = 1/r - \hat{d}$. In that case, the system just oscillates between $\hat{d}$ and $1/r - \hat{d}$; the planners payoff is

$$V(\hat{d}) = \frac{2\lambda}{r} R(\hat{d})$$

We show that in this case that V is differentiable at $\hat{d}$, implying that $V'(\hat{d}) = 0$ since V is flat to the left of $\hat{d}$, which means that the first order condition

$$R'(d_1) = -V'(d_1) + \mu(d)$$

cannot be satisfied if $d_1 = d = \hat{d}$, since the envelope condition would then imply

$$\begin{aligned} R'(d_1) &= -V'(d_1) + V'(d) \\ &= 0 \end{aligned}$$

To show that V is differentiable at $\hat{d}$, we describe a differentiable function $\tilde{V}$ that is below V near $\hat{d}$. Since V is concave, the existence of such a function implies that V is differentiable.

To construct $\tilde{V}$, suppose the planner delivers duration away from $\hat{d}$ by ε units by giving firm one extra duration at all future points when the other firm has the most recent innovation (and $x = 1$ when firm one has the most recent innovation). This implies that all innovations by firm 1 receive $\hat{d} + \frac{r}{\lambda}\varepsilon$, and all innovations by firm 2 receive $\hat{d} - \frac{r}{\lambda}\varepsilon$. Therefore the planner's payoff

$$\tilde{V}(\hat{d} + \varepsilon) = \frac{\lambda}{r} R(\hat{d} + \frac{r}{\lambda}\varepsilon) + \frac{\lambda}{r} R(\hat{d} - \frac{r}{\lambda}\varepsilon)$$

Under the maintained assumption that $V(\hat{d}) = \frac{2\lambda}{r} R(\hat{d})$, $\tilde{V}$ is a differentiable function equal to V at $\hat{d}$. Since it is feasible choice for the planner, must be less than the payoff V from the optimal policy. But therefore V is differentiable, implying that $d_1(\hat{d})$ must exceed $\hat{d}$, and contradicting that $V(\hat{d}) = \frac{2\lambda}{r} R(\hat{d})$. □

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